1901. Multiplying the brackets together before squaring,

$$\begin{aligned}
&(3^x+1)^2(3^{-x}-1)^2 \\
\equiv\ &(1-3^x+3^{-x}-1)^2 \\
\equiv\ &(3^{-x}-3^x)^2 \\
\equiv\ &9^x-2+9^{-x} \\
=\ &y-2+{}^1\!/\!_y.
\end{aligned}$$

1902. The best linear approximation to f at $x=\sqrt{8}$ is g such that $y=\mathrm{g}(x)$ is tangent to $y=\mathrm{f}(x)$. The derivative is $\mathrm{f}'(x)=3x^2-4$. The values are

$$\begin{aligned}
\mathrm{f}'\left(\sqrt{8}\right)&=20, \\
\mathrm{f}\left(\sqrt{8}\right)&=8\sqrt{2}.
\end{aligned}$$

Hence, the tangent line is $y-8\sqrt{2}=20\big(x-\sqrt{8}\big)$. Simplifying, $\mathrm{g}(x)=20x-32\sqrt{2}$.

1903. Substituting in, we have $\sin^2 t+\sin^2 2t=1$. The double-angle formula $\sin 2t\equiv 2\sin t\cos t$ gives

$$\begin{aligned}
&\sin^2 t+4\sin^2 t\cos^2 t=1 \\
\Longrightarrow\ &\sin^2 t+4\sin^2 t(1-\sin^2 t)=1 \\
\Longrightarrow\ &4\sin^4 t-5\sin^2 t+1=0 \\
\Longrightarrow\ &(4\sin^2 t-1)(\sin^2 t-1)=0 \\
\Longrightarrow\ &\sin t=\pm{}^1\!/\!_2,\pm 1.
\end{aligned}$$

The value $t={}^\pi\!/\!_6$ is a root of $\sin t={}^1\!/\!_2$. This lies in $[0,{}^\pi\!/\!_6]$, so the curve does intersect the circle.

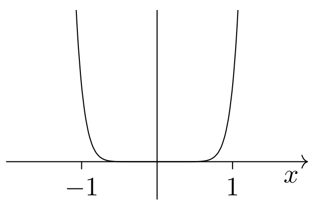
1904. In most statistical analyses, a statistician wants to represent the underlying population. In most types of distribution, there are more data around the centre, and fewer in the tails. This imbalance in the amount of data means:

- It is possible to make images with finer detail in the centre, because there are more data. This is achieved with narrow classes.
- It is necessary to address sampling variation in the tails, because there are fewer data. This is achieved with broad classes.

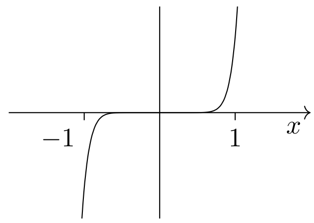
——— Nota Bene ———

One way of thinking about this is using the idea of *focus*. A histogram tends to be in sharper focus at the centre, bringing out detail, and blurrier in the tails, to cover up errors.

1905. (a) The even-powered curves have a broadly parabolic shape, which becomes more and more "snub-nosed" as the degree increases. So, for large $k\in\mathbb{N}$, $y=x^{2k}$ looks like



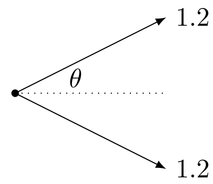
(b) The odd-powered curves have a broadly cubic shape, which hugs the x axis more tightly on the interval $(-1,1)$ as the degree increases. So, for large $k\in\mathbb{N}$, $y=x^{2k+1}$ looks like



1906. Integrating, the second derivative is $kx+c_1$, the first derivative is $\frac{1}{2}kx^2+c_1x+c_2$, and the original function is $\frac{1}{6}kx^3+\frac{1}{2}c_1x^2+c_2x+c_3$. Simplifying the constants, the set of solution curves is all cubics of the form

$$y=\tfrac{1}{6}kx^3+bx^2+cx+d.$$

1907. (a) Using triangle geometry, the two parts of the sling are at angle θ to the horizontal, where $\tan\theta=\frac{5}{20}=\frac{1}{4}$. Hence, our force diagram is



This gives NII horizontally as

$$\begin{aligned}
&2\times 1.2\cos\theta=0.04a \\
\Longrightarrow\ &a=58.2\text{ ms}^{-2}\text{ (3sf)}.
\end{aligned}$$

(b) In order to find the angle in part (a), we had to assume that the ball-bearing was a particle, i.e. that it had negligible size. We also assumed that the sling itself had negligible thickness.

1908. Substituting 75° into the identity,

$$\begin{aligned}
&1+\cot^2 75^\circ=\big(\sqrt{6}-\sqrt{2}\big)^2 \\
\Longrightarrow\ &\cot^2 75^\circ=7-4\sqrt{3} \\
\Longrightarrow\ &\tan^2 75^\circ=\tfrac{1}{7-4\sqrt{3}}.
\end{aligned}$$

We then rationalise the denominator, multiplying top and bottom by its conjugate $7+4\sqrt{3}$:

$$\begin{aligned}
\tan^2 75^\circ&=\frac{7+4\sqrt{3}}{49-48} \\
&=7+4\sqrt{3} \\
&=\big(2+\sqrt{3}\big)^2.
\end{aligned}$$

Taking the positive square root, $\tan 75^\circ=2+\sqrt{3}$, as required.

1909. Since the median is 0, the data are, in ascending order,

$$\{a, b, 0, c, 1\}.$$

The greatest value of a, and hence the least value of R_x, occurs when $a = b$ and $c = 0$. In this case, the mean gives $a = b = -\frac{1}{2}$ and $R_x = \frac{3}{2}$. The least value of a, and so the greatest value of R_x, occurs when $b = c = 0$, giving $a = -2$ and $R_x = 3$. But neither of these bounds is attainable, as each would have two modes, rather than simply 0 as stated in the question. Hence, the inequalities are strict, giving $\frac{3}{2} < R_x < 3$ as required.

1910. Writing 4^x as $(2^x)^2$, we can express the left-hand side's denominator as a difference of two squares. It factorises: $4^x - 1 \equiv (2^x - 1)(2^x + 1)$. Dividing top and bottom by $(2^x - 1)$ yields the required result.

1911. In each case, the rightwards implication obviously holds. So, the question is whether the leftwards implication holds or not.

(a) The implication is $x = y \implies \sin x = \sin y$. A counterexample to the leftwards implication is $x = 0, y = \pi$.

(b) Being an inverse, arcsin has to be one-to-one. So, $x = y \iff \arcsin x = \arcsin y$.

(c) The implication is $x = y \implies |x| = |y|$. A counterexample to the leftwards implication is $x = 1, y = -1$.

1912. We factorise top and bottom, finding the factors by spotting them or by solving $3x^2 + 5x + 2 = 0$ and $6x^2 - 11x - 10 = 0$. This gives

$$\frac{(3x + 2y)(x + y)}{(3x + 2y)(2x - 5y)} \equiv \frac{x + y}{2x - 5y}.$$

1913. Using (a, b, c) with c as the hypotenuse, the whole square has area c^2. This can also be expressed as the sum of four triangles and the central square, giving

$$\begin{aligned} &(b - a)^2 + 4 \times \tfrac{1}{2}ab = c^2 \\ \implies\ & b^2 - 2ab + 2ab + a^2 = c^2 \\ \implies\ & a^2 + b^2 = c^2, \text{ as required.} \end{aligned}$$

1914. This is true. Both the statements are expressions of the condition of independence. If information about B occurring doesn't affect the probability of A (as the left-hand statement says), then it cannot affect the probability of not-A (as the right-hand statement says). The reverse also holds.

1915. Any set of three distinct, concurrent lines provides a counterexample, such as $y = x$, $y = 2x$, $y = 3x$. These all meet at a unique point, $(0, 0)$.

1916. Set the first derivative to zero to find stationary points: $4x^3 - 9x^2 = 0$. This has solution $x = 0$ or $x = \frac{9}{4}$. The second derivative, then, is

$$\frac{d^2y}{dx^2} = 12x^2 - 18x = 6x(2x - 3).$$

This is zero at $x = 0$. Furthermore, there is only a single factor of x, so the second derivative changes sign at $x = 0$. Hence, this stationary point is also a point of inflection.

1917. (a) An arithmetic sequence is linear, and adding a linear function to a quadratic function yields another quadratic function. Hence, $A_n + B_n$ is quadratic.

(b) This is neither. It is cubic, in fact, since the ordinal formulae are polynomials of degree 2 and 1, and are multiplied together.

1918. (a) At $t = 0$, the positions are $x_1 = x_2 = \frac{0}{1} = 0$.

(b) Equating the positions, they are in the same place when

$$\begin{aligned} &\frac{t^2}{t^2 + 1} = \frac{4t}{4t + 1} \\ \implies\ & t^2(4t + 1) = 4t(t^2 + 1) \\ \implies\ & 4t^3 + t^2 = 4t^3 + 4t \\ \implies\ & t^2 - 4t = 0 \\ \implies\ & t = 0, 4 \end{aligned}$$

The root $t = 0$ gives the initial positions, so we want $t = 4$. Substituting gives $x_1 = x_2 = \frac{16}{17}$.

(c) In the long-term, both formulae give positions of the form $\frac{n}{n+1}$, which tend towards 1 as n gets large. Hence, both particles approach $x = 1$ as $t \to \infty$.

1919. The equation has roots when either factor is equal to zero. For the LH factor, this gives $x = -1, -2$. For the RH factor, $x^2 = -1, -2$. However, since x^2 is always positive, the biquadratic yields no real roots. Overall, the equation has 2 real roots.

Nota Bene

You can't use the discriminant here, for either of two reasons: some roots might be repeated across the two brackets, and the RH factor is a biquadratic, which may yield non-real roots even if the quadratic has non-negative discriminant.

1920. The y coordinate at $x = \frac{\pi}{4}$ is

$$y = 1 - \sin^2 \tfrac{\pi}{4} = 1 + \left(\tfrac{\sqrt{2}}{2}\right)^2 = \tfrac{3}{2}.$$

Next, using the chain rule,

$$\frac{dy}{dx} = 2 \sin x \cos x.$$

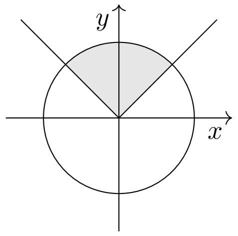
Evaluating this, $m = 1$. This gives the equation of the tangent as

$$y - \tfrac{3}{2} = 1(x - \tfrac{\pi}{4}).$$

Multiplying by four and rearranging the terms,

$$4y + \pi = 4x + 6, \text{ as required.}$$

1921. The interior of the circle is the possibility space here, so we need to calculate the relevant areas. The successful region is



The successful region is a quarter of the possibility space, so $p = \frac{1}{4}$.

1922. (a) This is true. At the root on the left, the graph is not tangential to the x axis. This root must, therefore, be a single root.

(b) This is true. Since the graph is cubic and the root on the left is a single root, the root on the right, at which there is no sign change, must be a double root.

(c) This is false.

(d) This is true. By definition, a double root is a repeated root.

1923. Completing the square, we have

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}. \end{aligned}$$

This is a transformation of the graph $y = x^2$: translation by vector $-\frac{b}{2a}\mathbf{i}$, followed by a stretch in the y direction with scale factor a, followed by a translation by vector $(c - \frac{b^2}{2a})\mathbf{j}$. This proves the result by construction.

1924. (a) The definitions of the sin and cos functions as the y coordinate and x coordinates of a point on the unit circle are symmetrical in the line $y = x$, which is $\theta = 45°$. The angles θ and $(90 - \theta)$ are equidistant from $\theta = 45°$. This proves the result.

(b) On a unit circle, adding 90° to θ rotates by a right angle, producing a perpendicular. Hence, the new gradient $\tan(90° + \theta)$ is the negative reciprocal of $\tan\theta$. This is $-\cot\theta$, as required.

1925. (a) Completing the square, $\mathrm{g}(x) = (x-1)^2 - 9$. Hence, the vertex of the parabola $y = \mathrm{g}(x)$ is at $(1, -9)$. Since the quadratic is positive, this means $\mathrm{g}(x)$ is increasing on $(1, \infty)$.

(b) The function $\mathrm{f}(x) = 10^x$ is exponential growth: it is increasing for all $x \in \mathbb{R}$. Hence, $\mathrm{fg}(x)$ is increasing exactly where $\mathrm{g}(x)$ is increasing, i.e. $x \in (1, \infty)$.

1926. We can distribute the differential operator $\frac{d}{dx}$ over the bracket:

$$\begin{aligned} &\frac{d}{dx}\left(x^3 + \frac{dy}{dx}\right) = 1 \\ \Longrightarrow\ & 3x^2 + \frac{d^2y}{dx^2} = 1 \\ \Longrightarrow\ & \frac{d^2y}{dx^2} = 1 - 3x^2. \end{aligned}$$

1927. Ignoring the green counters, we are choosing three locations for the red counters, giving a possibility space of ${}^6\mathrm{C}_3 = 20$ equally likely outcomes.

(a) There are two outcomes in which the reds (and therefore greens) are in a single row, which gives a probability of $\frac{2}{20} = 10\%$.

(b) There are also two outcomes in which the colours are in a chequerboard pattern, so the probability is again 10%.

1928. These are circles. The latter is the unit circle, and the former is

$$\left(x + \tfrac{3}{2}\right)^2 + \left(y + \tfrac{5}{2}\right)^2 = \tfrac{37}{2}.$$

This is centred on $\left(-\frac{3}{2}, -\frac{5}{2}\right)$, which is a distance of $\frac{\sqrt{34}}{2}$ from the origin. Therefore, since

$$\tfrac{\sqrt{34}}{2} = 2.91... < \sqrt{\tfrac{37}{2}} = 4.30...,$$

the origin lies inside the first circle. Furthermore, since the difference is greater than the radius of the unit circle, as in

$$1 + \tfrac{\sqrt{34}}{2} = 3.91... < \sqrt{\tfrac{37}{2}} = 4.30...,$$

the unit circle lies entirely within the first circle. Hence, they do not intersect, as required.

1929. This is false. Factorising the biquadratic,

$$\begin{aligned} y &= (x^2 - 4)(x^2 + 3) \\ &\equiv (x-2)(x+2)(x^2+3). \end{aligned}$$

This has no repeated factors, which rules out the possibility that $y = 0$ is tangent to the curve.

1930. The function is undefined when the denominator is zero. Since the index $2k$ is even and d is non-zero, this can only occur when $x^{2k} = c^{2k}$, which gives $x = \pm c$.

1931. This is a quadratic in $\operatorname{cosec} x$:

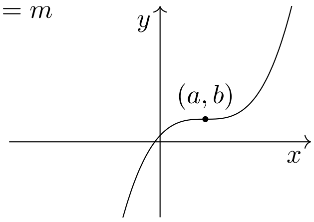
$$\begin{aligned} & \operatorname{cosec}^2 x + \operatorname{cosec} x - 2 = 0 \\ \implies & (\operatorname{cosec} x - 1)(\operatorname{cosec} x + 2) = 0 \\ \implies & \operatorname{cosec} x = 1, -2. \\ \implies & \sin x = 1, -\tfrac{1}{2}. \end{aligned}$$

There are three roots in total: $x = \frac{\pi}{2}$ from the first equation and $x = -\frac{\pi}{6}, -\frac{5\pi}{6}$ from the second.

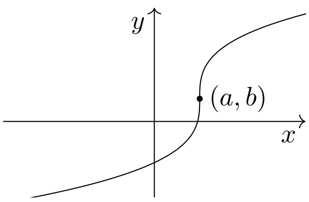
1932. A trivial counterexample is $\{0\}$, for which IQR and range are zero. A non-trivial counterexample is $\{0, 0, 0, 1, 1, 1\}$, for which both are 1.

1933. Each graph is an elementary cubic of the general shape of $y = x^3$, with its point of inflection at (a, b). The constant m determines the steepness of the cubic; this is drawn as $m = 1$ below.

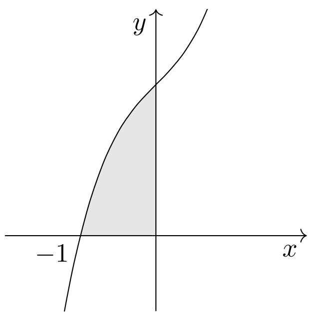
(a) $\dfrac{y - b}{(x - a)^3} = m$



(b) $\dfrac{(y - b)^3}{x - a} = m$



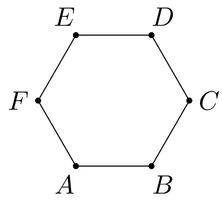
1934. We can factorise to $y = (x+1)(x^2-x+2) = 0$. The quadratic has negative discriminant, meaning that the curve has a single root at $x = -1$. Sketching, then, the required area is



So, the required area is

$$\begin{aligned} & \int_{-1}^{0} x^3 + x + 2 \, dx \\ = & \left[\tfrac{1}{4}x^4 + \tfrac{1}{2}x^2 + 2x\right]_{-1}^{0} \\ = & -\left(\tfrac{1}{4} + \tfrac{1}{2} - 2\right) \\ = & \tfrac{5}{4}. \end{aligned}$$

1935. The hexagon is



(a) Since $\overrightarrow{AB}$ has no $\mathbf{j}$ component, $\overrightarrow{CD}$ is the same as $\overrightarrow{BC}$ with the $\mathbf{i}$ component negated. This gives $\overrightarrow{CD} = -\mathbf{i} + \sqrt{3}\mathbf{j}$.

(b) Adding vectors tip-to-tail,

$$\begin{aligned} \overrightarrow{AD} &= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} \\ &= 2\mathbf{i} + (\mathbf{i} + \sqrt{3}\mathbf{j}) + (-\mathbf{i} + \sqrt{3}\mathbf{j}) \\ &= 2\mathbf{i} + 2\sqrt{3}\mathbf{j}. \end{aligned}$$

1936. Since inverse functions, in order to be invertible, must be one-to-one, the domain of each function must be the same set as the codomain of the other. Hence $A = D$ and $B = C$.

1937. Since x is a fixed point of the iteration, we know that $rx = x$. This gives $rx - x = 0$, which we can factorise to $x(r - 1) = 0$. Hence, $x = 0$ or $r = 1$.

1938. A counterexample is the function $f(x) = \frac{1}{x}$, with the values $x = -1$ and $x = 1$. There is a sign change, since $f(-1) = -1 < 0 < 1 = f(1)$, but there is no root between -1 and 1. Instead, the function is undefined at $x = 0$.

1939. The formula for conditional probability is

$$\mathbb{P}(X \mid Y) = \frac{\mathbb{P}(X \cap Y)}{\mathbb{P}(Y)}.$$

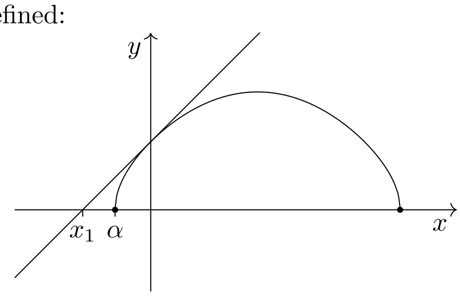
Substituting this in gives

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = k\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.$$

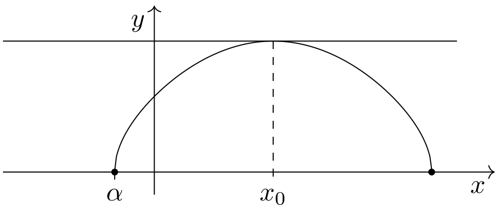
We can then divide by $\mathbb{P}(A \cap B)$, which is non-zero:

$$k = \frac{\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{0.4}{0.5} = 0.8.$$

1940. The Newton-Raphson method fails if it generates an approximation to the root which lies outside the domain of the function. For instance, if we look for the root of $f(x) = \sqrt{2\sin x + 1} = 0$ using the starting approximation $x_0 = 0$, we find that x_1 is not in the domain of the function, so x_2 is undefined:



A second type of failure occurs if x_0 is a stationary point. Then the tangent won't cross the x axis at all. So, using the same example as above, if we begin with $x_0 = \frac{\pi}{2}$, then x_1 is undefined:



A third type of semi-failure occurs if the gradient at x_0 is small, as it is in the vicinity of a stationary point. This could produce a tangent which finds a distant root, rather than a nearby one. This isn't a failure of the method as such, but rather a failure to find an intended root.

1941. The implication is $|x| = |y| \iff x^2 = y^2$. The modulus statement says that the magnitudes of the numbers x and y are the same, which means that the numbers are either equal or negatives of each other. Squaring x and $-x$ produces the same result. This gives three equivalent statements:

① $|x| = |y|$,
② $x = \pm y$,
③ $x^2 = y^2$.

1942. Both statements are false; the functions $\text{f}(x) = |x|$ and $\text{g}(x) = -x$ are counterexamples to both.

(a) $\text{f}'(-1) = -1$ but $|\text{g}'(-1)| = |-1| = 1$,
(b) $\int_{-1}^{1} \text{f}(x)\,dx = 1$, but $\left|\int_{-1}^{1} \text{g}(x)\,dx\right| = |0| = 0$.

1943. We can rewrite $y = 3^x$ over base 2. This produces

$$\begin{aligned} y &= 3^x \\ &\equiv \left(2^{\log_2 3}\right)^x \\ &\equiv 2^{x \log_2 3}. \end{aligned}$$

Comparing this to $y = 2^x$, we have replaced x by $x \log_2 3$. This is a stretch in the x direction, with scale factor $\frac{1}{\log_2 3} = \log_3 2$.

1944. Substituting in, we wish to show that

$$(n+1)^2 + (n+1) + 3 < 2(n^2 + n + 3)$$

for all n. Simplifying, this is $0 < n^2 - n + 1$. Since the RHS is a positive quadratic, we need only show that the associated equation $n^2 - n + 1 = 0$ has no real roots. Δ is $b^2 - 4ac = 1 - 4 = -3$, which proves that $u_{n+1} < 2u_n$ for all $n \in \mathbb{N}$.

1945. Replacing x by $(x - k)$ translates a graph by k in the positive x direction. So, the curves of odd degree have rotational symmetry around the point $(k, 0)$ and the curve of even degree has a line of reflective symmetry at $x = k$. Hence, the relevant equations are

(a) $y = -(x - k)$,
(b) $y = (x - k)^2$,
(c) $y = -(x - k)^3$.

1946. The LH expression is not well defined, as t is set to a: this makes the denominator of the fraction zero. In the RH expression, however, t is not equal to a, but only approaches arbitrarily close to it. And we can take a factor of $(t - a)$ out on the top and bottom, to get

$$\lim_{t \to a} \frac{(t-a)(t^2 + at + a^2)}{(t-a)(t+a)}.$$

Then, since $t \neq a$, we can divide top and bottom by $(t - a)$, giving

$$\lim_{t \to a} \frac{t^2 + at + a^2}{t + a}.$$

At this point, it is safe to take the limit, which, since a is non-zero, yields $\frac{1}{2a}\left(a^2 + a^2 + a^2\right) \equiv \frac{3}{2}a$.

1947. The first face can be chosen arbitrarily, wlog, since a dodecahedron is symmetrical. Then, since the two faces chosen are distinct, there remain 11 faces from which to choose the second one. Out of these, 5 border the original one. Hence, the probability is $5/11$.

1948. Every fifth natural number is divisible by 5. So, the first 125 natural numbers contain 25 numbers which are divisible by 5, hence 100 which are not divisible by 5. The sum of the first 125 is

$$S = \tfrac{1}{2} \times 125 \times 126 = 7875.$$

The sum of those divisible by 5 is

$$S = 5 \times \tfrac{1}{2} \times 25 \times 26 = 1625.$$

Subtracting these gives 6250, as required.

1949. Multiplying up by $x^2 - 1$,

$$\begin{aligned} &4x(x-1) - 4x^2(x+1) + 15(x^2 - 1) = 0 \\ \Longrightarrow\ &4x^3 - 15x^2 + 4x + 15 = 0. \end{aligned}$$

Since $x = 3$ is a root, the factor theorem tells us that $(x - 3)$ is a factor of the LHS. Taking this factor out,

$$(x-3)(4x^2 - 3x - 5) = 0.$$

Using the quadratic formula,

$$x = 3, \frac{3 \pm \sqrt{89}}{8}.$$

1950. Over the domain shown, the graph $y = \mathrm{f}(x)$ has negative gradient, which means the function f is decreasing.

The gradient f', while negative, is increasing on the domain shown, as the graph is getting shallower. So, the second derivative is positive, which means the function f is convex.

1951. Using the index laws $\left(a^b\right)^c \equiv a^{bc}$ and $a^{b+c} \equiv a^b a^c$,

(a) $e^{3x} = (e^x)^3$,

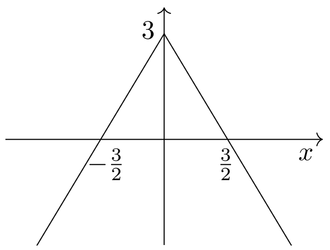
(b) $e^{3x-1} = \dfrac{(e^x)^3}{e}$,

(c) $e^{3x-2} = \dfrac{(e^x)^3}{e^2}$.

1952. As it stands, we cannot take the limit, i.e. send x to infinity, as $\frac{\infty}{\infty}$ isn't well defined. So, we divide top and bottom by x (which can be taken as non-zero, since x is heading for positive infinity). This allows us to take the limit, giving

$$\lim_{x\to\infty} \frac{2+\frac{5}{x}}{2-\frac{1}{x}} = \frac{2}{2} = 1.$$

1953. The output of $|x|$ is indeed always positive, but $3 - 2|x|$ involves multiplication by -1 (reflection in the x axis), which produces y values below zero. The graph is



1954. Vectors $x_1\mathbf{i} + y_1\mathbf{j}$ and $x_2\mathbf{i} + y_2\mathbf{j}$ are perpendicular if and only if

$$\frac{y_1}{x_1} = -\frac{x_2}{y_2}.$$

So, we require

$$\frac{5}{x} = \frac{x+3}{2}.$$

This is $x^2 + 3x - 10 = 0$, so $x = -5, 2$.

1955. The lengths of the perpendicular sides are given by $x = 3+t$ and $y = 4+t$. So, the area equation is $\frac{1}{2}(3+t)(4+t) = 15$. Solving this yields $t = -9, 2$. We need the positive $t = 2$.

The hypotenuse is

$$\begin{aligned} z &= \sqrt{(3+t)^2 + (4+t)^2} \\ &\equiv \sqrt{25 + 14t + 2t^2}. \end{aligned}$$

Differentiating by the chain rule,

$$\tfrac{1}{2}\left(25 + 14t + 2t^2\right)^{-\frac{1}{2}}(14 + 4t)\Big|_{t=2} = \frac{11}{\sqrt{61}}.$$

1956. (a) Since the two tail probabilities $\mathbb{P}(Y < 0)$ and $\mathbb{P}(Y > 4)$ are equal, the mean must lie halfway between 0 and 4, at $\mu = 2$.

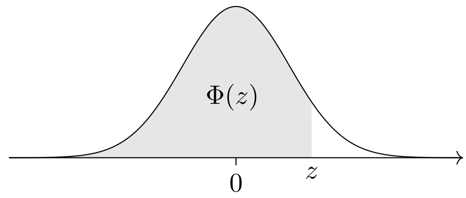
(b) $y = 4$ is $2/\sigma$ standard deviations above the mean. The probability up to this value is 80%. So,

$$\Phi^{-1}(0.8) = 0.8416 = 2/\sigma.$$

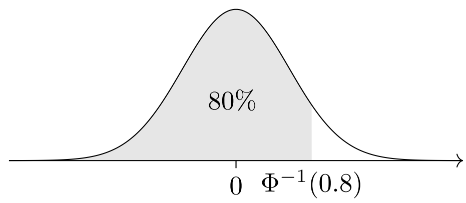
Hence, $\sigma = \frac{2}{0.8416} = 2.38$ (3sf).

— Nota Bene —

The function Φ is the area function of the normal distribution $Z \sim N(0,1)$. So, $\Phi(z)$ finds the area up to z:



The inverse Φ^{-1} calculates the z value from a given area/probability. With reference to $Z \sim N(0,1)$, the visual of the calculation above is



1957. (a) The cubic is not one-to-one over the domain $\mathbb{R}$. For instance, the equation $x^3 - 12x = 0$ has three roots at $x = 0, \pm\sqrt{12}$: three inputs map to the same output.

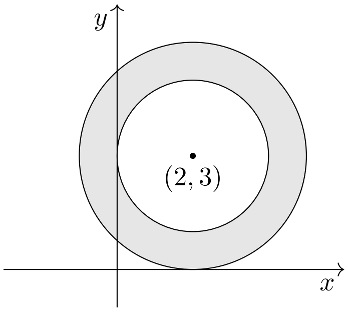
(b) Differentiating, $\mathrm{F}'(x) = 3x^2 - 12$. For SPs, $3x^2 - 12 = 0$, which has solution set $\{\pm 2\}$. These are turning points, so the domain of an invertible version of F cannot extend beyond them. Hence, the largest a is 2.

(c) Substituting $x = \pm 2$ back into the function gives coordinates $(\pm 2, \mp 16)$. So, the codomain for this version of F must be $[-16, 16]$.

1958. (a) If the inequality has this solution set, then the boundary equation has solution set $\{4, 5\}$. But there are infinitely many quadratic equations which have this solution set. Any quadratic of the form $a(x-4)(x-5) = 0$ does. Hence, a (and therefore b and c) cannot be determined using the information given.

(b) Since the solution set of the inequality is given as *outside* the values $x = 4$ and $x = 5$, the quadratic must be positive. So, $a \in (0, \infty)$.

1959. The boundary equations $(x-2)^2 + (y-3)^2 = 4, 9$ are a pair of concentric circles, each with centre $(2, 3)$ and with respective radii 2 and 3. Hence, the area we require is the annulus between the two:



1960. Differentiating by the chain rule and setting the derivative to zero, the function is stationary for

$$\begin{aligned} &\cos x - 2\cos x \sin x = 0 \\ \Longrightarrow\ &\cos x(1 - 2\sin x) = 0 \\ \Longrightarrow\ &\cos x = 0 \text{ or } \sin x = \tfrac{1}{2}. \end{aligned}$$

Over $[0, 2\pi)$, this gives

$$x = \tfrac{\pi}{2}, \tfrac{3\pi}{2}, \tfrac{\pi}{6}, \tfrac{5\pi}{6}.$$

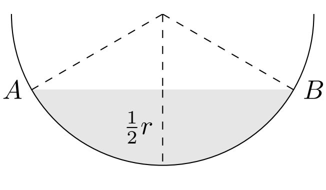
Using a double-angle formula, the first derivative is $\cos x - \sin 2x$, Hence, the second derivative is $-\sin x - 2\cos 2x$. This is positive at $\pi/2, 3\pi/2$, and negative at $x = \pi/6, 5\pi/6$. So, the function has local maxima at $x = \pi/6, 5\pi/6$.

1961. (a) By symmetry, the probability that a variable is positive is 0.5. Since the two are independent, $\mathbb{P}(Z_1, Z_2 > 0) = 0.25$.

(b) The probability that $Z_1 = Z_2$ is zero, since the normal distribution is continuous. Hence, it is equally likely that either variable is larger: $\mathbb{P}(Z_1 < Z_2) = 0.5$.

(c) Since we need both variables to be positive, and we also need $Z_1 < Z_2$, we multiply the answers to the first two parts. This gives $\mathbb{P}(0 < Z_1 < Z_2) = 0.125$.

1962. The semicircular cross-section looks as follows:



Angle AOB is 120°. Hence, sector OAB has area $\frac{1}{3}\pi r^2$. Then, triangle OAB has area $\frac{1}{2}r^2 \sin 120°$, which is $\frac{1}{4}\sqrt{3}r^2$. So, the area of the segment is $\frac{1}{3}\pi r^2 - \frac{1}{4}\sqrt{3}r^2$. Hence, to obtain the fraction we require, we divide by the area of the semicircle, which is $\frac{1}{2}\pi r^2$. This gives

$$\frac{\frac{1}{3}\pi r^2 - \frac{\sqrt{3}}{4}r^2}{\frac{1}{2}\pi r^2} = \tfrac{1}{6}\big(4\pi - 3\sqrt{3}\big).$$

1963. (a) With $n = -1$ and $4x$ in place of x, we take the first three terms of the binomial expansion:

$$\begin{aligned} (1 + 4x)^{-1} &\approx 1 + (-1)(4x) + \frac{(-1)(-2)}{2!}(4x)^2 \\ &\equiv 1 - 4x + 16x^2. \end{aligned}$$

(b) The expansion converges for $|4x| < 1$, which we can rewrite as $|x| < \frac{1}{4}$.

1964. We rearrange as follows:

$$\begin{aligned} &(3x-2)^3 + 8 = 12x \\ \Longrightarrow\ &(3x-2)^3 - 4(3x-2) = 0 \\ \Longrightarrow\ &(3x-2)\big((3x-2)^2 - 4\big) = 0. \end{aligned}$$

The latter factor is a difference of two squares:

$$\begin{aligned} &(3x-2)(3x-2+2)(3x-2-2) = 0 \\ \Longrightarrow\ &(3x-2)(3x)(3x-4) = 0 \\ \Longrightarrow\ &x = 0, \tfrac{2}{3}, \tfrac{4}{3}. \end{aligned}$$

1965. Using $\sin^2\theta + \cos^2\theta \equiv 1$, the LHS is $\cos^2\theta = \frac{1}{4}$. The RHS is the positive square root of this. Hence, the first implication doesn't hold, as the negative root is also an option. The second implication does hold, as can be shown by squaring.

1966. We know that $l = r\theta$, with θ in radians. Hence, since the angle is changing by ω rad/s, the arc length l is changing by $r\omega$ units/s. This gives $v = r\omega$.

— Nota Bene —

This formula can also be derived by differentiating both sides of $l = r\theta$ with respect to time.

1967. (a) Pick a triangle: there are two colourings. From here, each pair of neighbouring triangles must be coloured differently, so this leaves no further options. This gives two ways overall.

(b) There are no restrictions here, so each triangle may be coloured in one of three ways. There are nine triangles, giving $3^9 = 19683$ ways.

1968. The associated equation $x^2 - x + 6 = 0$ has no real roots, as its discriminant is $\Delta = -23$. This puts the parabola $y = x^2 - x + 6$ entirely above the x axis. Hence, the inequality is satisfied by any real x: the solution set is $\mathbb{R}$.

1969. The parabolae are reflections in $y = x$. So, if one intersects $y = x$, then they intersect each other. Solving simultaneously to find these intersections, $x^2 - x = x$, so $x = 0$ or $x = 2$.

The first derivative is $\frac{dy}{dx} = 2x - 1$. Evaluating this at $x = 0$, we get -1. So, the tangent is $y = -x$. Since this line crosses $y = x$ at right angles, it must also be tangent to $x = y^2 - y$. Hence, they are tangent to each other.

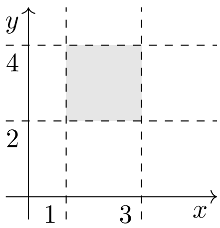
1970. We multiply by 10^n, shifting the digits by n:

$$\begin{aligned} x &= 0.a_1a_2...a_na_1a_2... \\ 10^n x &= \quad a_1a_2...a_na_1a_2... \end{aligned}$$

Subtracting the upper line from the lower,

$$\begin{aligned} (10^n - 1)x &= a_1a_2...a_n \\ \implies x &= \frac{a_1a_2...a_n}{\underbrace{99...9}_{n \text{ 9's}}}, \text{ as required.} \end{aligned}$$

1971. The associated equations give the pairs of lines $x = 1, 3$ and $y = 2, 4$. The region satisfying both inequalities is the square bordered by these lines:



1972. Integrating,

$$\begin{aligned} s &= \int_0^4 3t^2 + 2\,dt \\ &= \Big[t^3 + 2t\Big]_0^4 \\ &= 72. \end{aligned}$$

So, the average velocity over the time period is $\bar{v} = \frac{72}{4} = 18$. Therefore, we want $3t^2 + 2 = 18$, which gives $t = \pm 4/\sqrt{3}$. Only the positive time lies in $[0, 4]$, so $t = 4/\sqrt{3}$.

1973. We factorise to $(1+x)^5(1+2x)^5$. Expanding each of the factors up to the term in x^2 gives

$$(1 + 5x + 10x^2 + ...)(1 + 10x + 40x^2 + ...).$$

When multiplying out, there are three terms in x^2: $40x^2 + 50x^2 + 10x^2$. So, the coefficient is 100.

1974. Doubling the second equation, we have

$$\begin{aligned} 2\sin x + \cos y &= 1, \\ 2\sin x - 8\cos y &= 10. \end{aligned}$$

The difference is $9\cos y = -9$, so $\cos y = -1$, which gives $\sin x = 1$. Each equation has one root in the given domain: (x, y) is $(90°, 180°)$.

1975. The normal distribution will not apply here, for a number of reasons. Two are as follows:

① $N(\mu, \sigma^2)$ is unimodal. Bu there are male and female subpopulations, so the population is likely to show some bimodality.

② $N(\mu, \sigma^2)$ is symmetrical about the mean. But the population is certain to be skewed, given that the upper bound (world-record speed) is significantly closer to the mean that the lower bound (speed zero).

1976. Since $a \neq b$, $c \neq d$,

$$\begin{aligned} & \frac{a+b}{a-b} = \frac{c+d}{c-d} \\ \iff & (a+b)(c-d) = (a-b)(c+d) \\ \iff & ac - ad + bc - bd = ac + ad - bc - bd \\ \iff & -ad + bc = ad - bc \\ \iff & bc = ad \end{aligned}$$

The implication running up the page is the one we need. We can now restart with the first equation:

$$\frac{a}{b} = \frac{c}{d} \implies bc = ad.$$

Putting these two implications together gives the required result.

1977. Solving $\sqrt{x} = x$ gives $x = 0, 1$, so the domain is $[0, 1]$. In this domain, the square root graph is above $y = x$, so the height difference between the curves is given by $\text{f}(x) = \sqrt{x} - x$. Hence, the true area enclosed is

$$\begin{aligned} I &= \int_0^1 x^{\frac{1}{2}} - x\,dx \\ &= \left[\tfrac{2}{3}x^{\frac{3}{2}} - \tfrac{1}{2}x^2\right]_0^1 \\ &= \tfrac{1}{6}. \end{aligned}$$

The trapezium rule approximation with four strips has strip width $h = \frac{1}{4}$. Hence, it is given by

$$\begin{aligned} T &= \tfrac{1}{2} \cdot \tfrac{1}{4}\Big\{\text{f}(0) + 2\Big(\text{f}(\tfrac{1}{4}) + \text{f}(\tfrac{1}{2}) + \text{f}(\tfrac{3}{4})\Big) + \text{f}(1)\Big\} \\ &= 0.143283... \end{aligned}$$

The percentage error is

$$\begin{aligned} \frac{T - I}{I} &= \frac{0.143283... - 1/6}{1/6} \\ &= -0.1403.. \end{aligned}$$

Hence, the trapezium rule underestimates the true value of the area by approximately 14%.

1978. Carrying out the integrals,

$$\begin{aligned} \int \frac{1}{x^2}\,dx &= \int \frac{1}{y^2}\,dy \\ \implies -x^{-1} + c_1 &= -y^{-1} + c_2 \end{aligned}$$

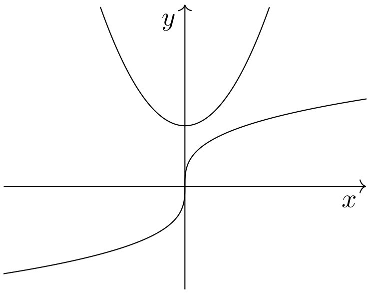
We multiply by -1, and combine the constants into a single k, choosing the x side since we want y as the subject. This gives $y^{-1} = x^{-1} + k$. Taking the reciprocal of both sides,

$$y = \frac{1}{x^{-1} + k}.$$

Multiplying top and bottom by x gives

$$y = \frac{x}{kx + 1}.$$

1979. Consider $x = y^3$. To produce a parabola that does not intersect with this, we translate $y = x^2$ in the y direction. So, a counterexample is $x = y^3$ and $y = x^2 + 1$:



——— Nota Bene ———

If the quadratic and cubic both had the form $y = \mathrm{f}(x)$, then they would have to intersect. The equation for intersections would be a cubic, and every cubic equation has at least one real root.

In the counterexample, however, the equation for intersections is a sextic: $x = (x^2 + 1)^3$. A sextic has even degree, and can have no real roots.

1980. (a) The function is quadratic in 2^x. Let $z = 2^x$, and it is $\mathrm{g}(z) = 4z^2 - 5z + 1$. Completing the square gives

$$\mathrm{g}(z) = 4(z - 5/8)^2 - 9/16.$$

So, since $z = 2^x$ takes all values in $\mathbb{R}^+$, this has minimum $-9/16$ and range $\{y \in \mathbb{R} : y \geq -9/16\}$.

(b) With $z = 2^x$, we factorise: $(4z - 1)(z - 1) = 0$. So, $2^x = 1, 1/4$. Taking logs, $x = 0, -2$. These differ by 2, as required.

1981. Performing the implicit differentiation $\frac{d}{dx}$ using the chain rule,

$$\begin{aligned} &\tfrac{d}{dx}\sqrt{x+y} = x \\ \Longrightarrow\ &\tfrac{1}{2}(x+y)^{-\frac{1}{2}}\left(1 + \tfrac{dy}{dx}\right) = x \\ \Longrightarrow\ &1 + \tfrac{dy}{dx} = 2x\sqrt{x+y} \\ \Longrightarrow\ &\tfrac{dy}{dx} = 2x\sqrt{x+y} - 1. \end{aligned}$$

1982. (a) Differentiating by the chain and quotient rules,

$$\begin{aligned} \mathrm{f}'(x) &= -e^{-x} + \frac{\frac{1}{x}\cdot x - \ln x}{x^2} \\ &\equiv -e^{-x} + \frac{1 - \ln x}{x^2}. \end{aligned}$$

This gives $\mathrm{f}'\left(\frac{1}{2}(1+\sqrt{5})\right) = -0.000129$ (3sf).

(b) The gradient is negative, so $x_0 = \frac{1}{2}(1+\sqrt{5})$ must be to the right of the stationary point. The tangent at x_0 will therefore cross the x axis at $x_1 > x_0$. And the same will be true of subsequent iterations: $x_{n+1} > x_n$. Hence, if $x_0 \geq \frac{1}{2}(1+\sqrt{5})$, the N-R iteration will diverge $x_n \to +\infty$, failing to find the root.

1983. By the definitions of sin and cos, and Pythagoras, we know that $\sin^2\theta + \cos^2\theta \equiv 1$. Dividing both sides by $\sin^2\theta$,

$$1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}.$$

By definition, cosec and cot are the reciprocals of sin and tan. So, $\cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$. QED.

——— Nota Bene ———

Should you go into a detailed proof of the first Pythagorean identity $\sin^2\theta + \cos^2\theta \equiv 1$ here? By convention, no. I've given a quick nod to such a proof above, but there's no point in always going right back to square one. In general, a good proof *quotes* results simpler than the one in question, pointing them out but not proving them.

1984. (a) Let $z = x^2 + 4$. Differentiating this equation with respect to x and then multiplying by 3,

$$\begin{aligned} \frac{dz}{dx} &= 2x \\ \Longrightarrow 3\,dz &= 6x\,dx. \end{aligned}$$

(b) Rearranging a little and making the variable in the limits explicit,

$$\begin{aligned} I &= \int_0^1 6x\sin(x^2+4)\,dx \\ &= \int_{x=0}^{x=1} \sin(x^2+4) \times 6x\,dx. \end{aligned}$$

The limits $x = 0, 1$ transform to $z = 4, 5$. We can now enact the substitution:

$$\begin{aligned} I &= \int_{z=4}^{z=5} \sin z \times 3\,dz. \\ &= 3\int_4^5 \sin z\,dz, \text{ as required.} \end{aligned}$$

——— Nota Bene ———

Writing a line like $3\,dz = 6x\,dx$ divides opinion in mathematical circles. Some mathematicians argue that you shouldn't write such things, because dx and dz don't mean anything on their own. And I agree, up to a point. To write dx on its own, *without the presence of another infinitesimal*, as in "$dx = x + 3$", is indeed nonsense. However, in $3\,dz = 6x\,dx$, the infinitesimals dz and dx *are* in the presence of another, albeit on the other side of the equation.

The equation $3\,dz = 6x\,dx$ expresses the limit of the relationship $3\,\delta z \approx 6x\,\delta x$ as the small finite quantities δz and δx approach zero. And that is a well defined idea. The point is, we can *choose* to give the equation $3\,dz = 6x\,dx$ exactly the same meaning as the equation $3\frac{dz}{dx} = 6x$. And, since the former gives significant benefit in terms of ease of use, I see it as a sensible thing to write.

1985. Applying f twice,

$$\begin{aligned} f^2(x) &= f\left(\frac{1}{1+x}\right) \\ &\equiv \frac{1}{1+\frac{1}{1+x}} \\ &\equiv \frac{1+x}{1+x+1} \\ &\equiv \frac{1+x}{2+x}. \end{aligned}$$

Hence,

$$\begin{aligned} f^3(x) &= f\left(\frac{1+x}{2+x}\right) \\ &\equiv \frac{1}{1+\frac{1+x}{2+x}} \\ &\equiv \frac{2+x}{2+x+1+x} \\ &\equiv \frac{2+x}{3+2x}, \text{ as required.} \end{aligned}$$

1986. Since AB is the diagonal of a unit square, its length is $\sqrt{2}$. The same is true of AC and BC. So, the triangle is equilateral. Using $\frac{1}{2}ab\sin C$, its area is

$$\begin{aligned} A_\triangle &= \tfrac{1}{2}\left(\sqrt{2}\right)^2 \sin 60° \\ &= \tfrac{\sqrt{3}}{2}. \end{aligned}$$

1987. (a) After the swap, A can have a pair of Jacks or Queens. The former requires A to give up the Queen, with probability $\frac{1}{3}$; the latter requires A to give up a Jack and receive a Queen, with probability $\frac{2}{3}\times\frac{2}{3}$. So,

$$\mathbb{P}(\text{A has a pair}) = \tfrac{1}{3} + \tfrac{2}{3}\times\tfrac{2}{3} = \tfrac{7}{9}.$$

(b) If B gives up the King, then B has at least a pair of Queens; this has probability $\frac{1}{3}$. If B gives up a Queen, then A must replace it; this has probability $\frac{2}{3}\times\frac{1}{3}$. So,

$$\mathbb{P}(\text{B has at least a pair}) = \tfrac{1}{3} + \tfrac{2}{3}\times\tfrac{1}{3} = \tfrac{5}{9}.$$

1988. Differentiating, we have $g'(x) = \frac{1}{2}x^{-\frac{1}{2}}$. We can now substitute $g(x)$ and $g'(x)$ into the LHS:

$$\begin{aligned} & 2\,g'(x)(g(x)-1) \\ \equiv\ & 2\cdot\tfrac{1}{2}x^{-\frac{1}{2}}\left(x^{\frac{1}{2}}+1-1\right) \\ \equiv\ & x^{-\frac{1}{2}}x^{\frac{1}{2}} \\ \equiv\ & 1. \end{aligned}$$

Hence, $g(x) = \sqrt{x}+1$ satisfies the relationship.

1989. Multiplying out the left-hand equation,

$$\begin{aligned} & \big(|x|-|y|\big)\big(|x|+|y|\big) = 0 \\ \Longleftrightarrow\ & |x|^2 - |y|^2 = 0 \\ \Longleftrightarrow\ & |x|^2 = |y|^2. \end{aligned}$$

Since a square is positive, we know that $|x|^2 \equiv x^2$ and $|y|^2 \equiv y^2$, which proves the result.

1990. The first term of the series is

$$\begin{aligned} & \frac{2\sqrt{2}}{9801}\,\frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}\bigg|_{k=0} \\ =\ & \frac{2\sqrt{2}}{9801}\,\frac{0!\times 1103}{(0!)^4 396^0} \\ =\ & 0.318309878... \end{aligned}$$

Reciprocating this gives 3.141592730..., while the true value of π is 3.1415926535.... The first six decimal places are correct, as required.

1991. The average value is given by

$$\begin{aligned} \overline{f(x)} &= \tfrac{1}{81}\int_0^{81} f(x)\,dx \\ &= \tfrac{1}{81}\int_0^{81} x^{\frac{1}{2}}\,dx \\ &= \tfrac{1}{81}\left[\tfrac{2}{3}x^{\frac{3}{2}}\right]_0^{81} \\ &= 6. \end{aligned}$$

1992. Since a binomial distribution is discrete, we can rewrite the LHS as $\mathbb{P}(X=0 \mid X=0,1)$. Using the formula for condition probability, and since $\{0\}$ is a subset of $\{0,1\}$, this is

$$\frac{\mathbb{P}(X=0)}{\mathbb{P}(X=0,1)} = \tfrac{1}{6}.$$

The binomial distribution formula then gives

$$\frac{(1-p)^5}{(1-p)^5+5p(1-p)^4} = \tfrac{1}{6}.$$

Dividing top and bottom by $(1-p)^4$,

$$\frac{1-p}{1-p+5p} = \tfrac{1}{6}.$$

Solving this yields $p = \frac{1}{2}$.

1993. (a) Using $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$, we get $\dfrac{dy}{dx} = \dfrac{3t^2}{6t} \equiv \dfrac{t}{2}$.

(b) When the particle is travelling parallel to the line $y = x$, $\frac{dy}{dx} = 1$, hence $\frac{t}{2} = 1$, giving $t = 2$. Substituting this into the expressions for x and y, the coordinates are $(8,7)$.

1994. Differentiating,

$$\begin{aligned} & y = x^3 - 4x^2 + 2x - 1 \\ \Longrightarrow\ & \frac{dy}{dx} = 4x^2 - 8x + 2 \\ \Longrightarrow\ & \frac{d^2y}{dx^2} = 8x - 8. \end{aligned}$$

At $x = 3$, the second derivative is $16 > 0$. The curve is polynomial, so the second derivative must remain positive for x values close to 3. Hence, the curve is convex in the vicinity of $(3,-4)$.

1995. The difference between the sequences is

$$a_n - b_n = -500 + 60n - 2n^2.$$

This is a negative quadratic, which has a maximum at $(15, -50)$. At this point, the value of $|a_n - b_n|$ is 50; at all other points, the value of $|a_n - b_n|$ will be greater. Hence, the minimum of $|a_n - b_n|$ occurs at $n = 15$.

1996. This is true. The second statement is simply the derivative with respect to x of the first. Hence, if the first statement holds, then so does the second.

1997. Force C exerts no moment around the centre, since the line of action goes through the centre. Forces B and E cancel out, since their lines of action are equidistant (horizontally to the left and right) from the centre. So, we need consider only forces D and A, which exert moments in opposite wise.

Since the forces have the same magnitude, only the perpendicular distance from the centre to the line of action matters. This is l for A and $\frac{\sqrt{3}}{2}l < l$ for D. Hence, A will exert a greater moment, and the merry-go-round will rotate anticlockwise.

1998. Let $x = \sqrt{y}$. This gives

$$\frac{x}{1+x} + \frac{x+1}{x-1} = 4.$$

Multiplying up by the denominators,

$$\begin{aligned} & x(x-1) + (x+1)^2 = 4(x-1)(x+1) \\ \Longrightarrow\ & x^2 - x + x^2 + 2x + 1 = 4x^2 - 4 \\ \Longrightarrow\ & 0 = 2x^2 - x - 5 \\ \Longrightarrow\ & x = \frac{1 \pm \sqrt{41}}{4}. \end{aligned}$$

Taking the positive root,

$$\begin{aligned} & \sqrt{y} = \frac{1 + \sqrt{41}}{4} \\ \Longrightarrow\ & y = \frac{21 + \sqrt{41}}{8}. \end{aligned}$$

1999. This is incorrect. When the accelerator is pressed, a driving force is indeed generated, but it acts backwards on the ground. The Newton III pair of this force, which is by definition also a driving force, acts forwards on the car. This is what causes the car to accelerate.

2000. The derivative of $\cot x$ is $-\operatorname{cosec}^2 x$. Substituting this into the LHS of the differential equation gives $\cot^2 x - \operatorname{cosec}^2 x + 1$. By the third Pythagorean trig identity, this is equal to zero, as required.

——— End of Volume II ———